Notes on the LM-KTH quasigeoid and geoid computations

The quasigeoid and geoid models are computed using a two-step procedure. First, the terrestrial land and debiased airborne gravity anomalies are gridded using a Remove-Compute-Restore technique and three-dimensional Least Squares Collocation. This results in a *surface* gravity anomaly grid. This step achieves downward continuation of the airborne data and combination with the terrestrial land observations. In the second step, this grid is used to compute the height anomaly and geoid height using Least Squares Modification of Stokes' formula with Additive corrections (LSMSA or KTH method). The *combined EGM* Cnm_refB_v050317a_s2-2190zt_4 with M=2190 is used in the first gridding step, while the *satellite-only EGM* GOCO05S with M=240 is used in step 2.

GRS 80 has consistently been used as normal gravity field, but the height anomaly and geoid heights are converted to refer to the conventional IHRS W_0 value 62 636 853.4 m^2s^{-2}. The non-tidal permanent tide system is used everywhere.

The potential value along the GSVS17 profile is finally derived using the generalized Bruns' formula (Eq. 2-178 in Heiskanen and Moritz 1967):

$$W_P = U_P + \gamma_{GRS80} \cdot \zeta_P + \left(W_0 - U_{0,GRS80}\right), \tag{2}$$

which has also been checked by comparing with the following (identical results),

$$\begin{aligned} H_P^N &= h_P - \zeta_P \\ C_P &= H_P^N \cdot \bar{\gamma}_{GRS80} \\ W_P &= W_0 - C_P \end{aligned} \tag{3}$$

Step 1: Surface gravity anomaly gridding using Least Squares Collocation

The surface gravity anomaly grid is computed using standard collocation with the following remove-compute-restore formula (Moritz 1980),

$$\Delta g_{surface} = C_{st}\left(C_{tt} + D_{tt}\right)^{-1} \begin{pmatrix} \Delta g_{land} - \Delta g_{EGM} - \Delta g_{RTM} - \Delta g_{ATM} - const_{terr} \\ \Delta g_{airborne} - \Delta g_{EGM} - \Delta g_{RTM} - \Delta g_{ATM} - const_{air} \end{pmatrix} + \Delta g_{EGM} + \Delta g_{RTM} + \Delta g_{ATM} + const_{air}$$

(4)

- The EGM (in this step 1) is Cnm_refB_v050317a_s2-2190zt_4 with maximum degree *M* = 2190, converted to the non-tidal system.
- The **debiased** 1 Hz airborne data in 'GRAVD_ms05_median_debiased_1hz.txt' is used. First down-sampled to 3' (0.05 degrees) and then converted from gravity disturbances to gravity anomalies using the above EGM with M = 2190.
- The RTM correction was computed using rectangular prisms (TC) and a mean reference surface computed by moving averaging in TCGRID (corresponding to *M* above). A 3'' x 3'' DEM is used in the remove step, while a DEM with the same resolution as for the gravity anomaly grid (1' x 1') is used in the restore step. This amounts to some smoothing of the topography. The indirect effect of this smoothing is corrected after step 2 (it is small, within +-1.2 cm everywhere).
- The final surface gravity anomaly grid extends the geoid target area with 2 degrees in latitude and 3 degrees in longitude.
- The resolution of the surface gravity anomaly grid is 0.5'x0.5'.
- The atmospheric correction is computed according to Moritz (2000).
- Signal covariance function by first computing an empirical covariance function for the reduced terrestrial land gravity anomalies and then fitting the following Tscherning and Rapp (1974) covariance function,

$$C(P,Q) = \sum_{n=2}^{21600} c_n \left(\frac{R^2}{r_P r_Q} \right)^{n+2} P_n\left(\cos\psi_{PQ}\right) \qquad (5a)$$

$$c_n = \sigma^2_{n,\,\mathrm{EGM}} \qquad n \leq 2160 \qquad (5b)$$

$$c_n = \alpha \frac{(n-1)}{(n-2)(n+4)} \left(\frac{(R-D)^2}{R^2} \right)^{n+2} \qquad n > 2160 \qquad (5c)$$

  The error degree variances for the EGM (Cnm_refB_v050317a_s2-2190zt_4), i.e. $\sigma^2_{n,\,\mathrm{EGM}}$ in Eq. (5b), are constructed based on GOCO05S and EGM2008 (with the EGM2008 coefficient standard errors divided by 2).
- The error covariance functions for the terrestrial land and airborne gravity anomaly data are constructed assuming uncorrelated (white) noise.

  Land: White noise: σ = 2 mGal

  Airborne: White noise: σ = 1.5 mGal
- Different constants are removed from the reduced land and airborne gravity anomalies in order to center them (i.e. to get zero means). The airborne constant is restored.
- The computation has been made by dividing the area into 1x1-degree blocks. Gravity anomalies covering the blocks with an overlap of 2 degrees in each direction are then picked out for LSC over each block. Finally, the blocks are merged and the differences along the block borders are checked.

Step 2: Height anomalies and geoid heights by the LSMSA (or KTH) method

The method if Least Squares Modification with Additive corrections; see e.g. Sjöberg (1991, 2003b) and Ågren et al. (2009) for details about the practical method. Stochastic kernel modification method of Sjöberg (1991). For the geoid height we have,

$$N = \frac{\left(GM_{GGM} - GM_{\mathrm{GRS\,80}}\right)}{R \cdot \gamma} - \frac{\left(W_{0,IHRS} - U_{\mathrm{GRS\,80}}\right)}{\gamma} + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^M(\psi)\Delta g_{surface} d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{M}\left(s_n + Q_n^M\right)\Delta g_n^{GGM} + \delta N_{COMB} + \delta N_{DWC} + \delta N_{ATM} + \delta N_{ELL}$$

$S^M(\psi)$ is the modified Stokes' function chosen according to Sjöberg (1991).
δN_{COMB} is the combined topographic effect (Sjöberg 2007)
δN_{DWC} is the total downward continuation correction; cf. Sjöberg (2003a) and Ågren (2004). Please see below for the computation of $\delta \mathrm{N}_{DWC}$
N_{ATM} is the atmospheric correction (Sjöberg and Nahavandchi 2000).
δN_{ELL} is the ellipsoidal correction (Sjöberg 2004).

(6)

where

$$\begin{aligned}
\delta N_{DWC} &= \frac{\Delta g(P)}{\gamma} H_P + 3\frac{\zeta_P^0}{r_P} H_P - \frac{1}{2\gamma}\left.\frac{\partial \Delta g}{\partial r}\right|_P H_P^2 + \\
&+ \frac{R}{2\gamma}\sum_{n=2}^{M}\left(s_n + Q_n^M\right)\left[\left(\frac{R}{r_P}\right)^{n+2} - 1\right]\Delta g_n^{GGM}(P) + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^M(\psi)\left(\left.\frac{\partial \Delta g}{\partial r}\right|_Q \left(H_P - H_Q\right)\right) d\sigma_Q \approx \\
&\approx \frac{\Delta g(P)}{\gamma} H_P + 3\frac{\zeta_P^0}{r_P} H_P + \\
&+ \frac{R}{2\gamma}\sum_{n=2}^{M}\left(s_n + Q_n^M\right)\left[\left(\frac{R}{r_P}\right)^{n+2} - 1\right]\Delta g_n^{GGM}(P) + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^M(\psi)\left(\left.\frac{\partial \Delta g}{\partial r}\right|_Q \left(H_P - H_Q\right)\right) d\sigma_Q
\end{aligned}$$

(7)

For the height anomaly,

$$\zeta = \frac{\left(GM_{GGM} - GM_{\mathrm{GRS\,80}}\right)}{r_P \cdot \gamma_Q} - \frac{\left(W_{0,IHRS} - U_{\mathrm{GRS\,80}}\right)}{\gamma_Q} + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^M(\psi)\Delta g_{surface} d\sigma + \frac{R}{2\gamma}\sum_{n=2}^{M}\left(s_n + Q_n^M\right)\Delta g_n^{GGM} + \zeta_{DWC} + \delta\zeta_{ATM} + \delta\zeta_{ELL}$$

$S^M(\psi)$ is the modified Stokes' function chosen according to Sjöberg (1991).
$\delta\zeta_{DWC}$ includes analytical continuation to point-level of both the gravity anomalies (Moritz 1980) and the spherical harmonic expansion; cf. Sjöberg (2003a) and Ågren (2004).
$\delta\zeta_{ATM}$ is the atmospheric correction (Sjöberg and Nahavandchi 2000).
$\delta\zeta_{ELL}$ is the ellipsoidal correction (Sjöberg 2004).

(8)

where

$$\delta\zeta_{DWC} = 3\frac{\zeta_P^0}{r_P}H_P + \frac{R}{2\gamma}\sum_{n=2}^{M}\left(s_n + Q_n^M\right)\left[\left(\frac{R}{r_P}\right)^{n+2} - 1\right]\Delta g_n^{GGM}(P) + \frac{R}{4\pi\gamma}\iint_{\sigma_0} S^M(\psi)\left(\left.\frac{\partial\Delta g}{\partial r}\right|_Q \left(H_P - H_Q\right)\right)d\sigma_Q \tag{9}$$

- The total downward continuation correction propagated to the geoid height, δN_{DWC}, is computed without the gradient term as indicated in Eq. (7). This means that a linear relation is assumed between the free air anomaly and the height (see Ågren 2004, Subsection 5.4.1). If the gradient term is included with the present rough/high terrain and resolution, very large oscillations are obtained (up to 0.5 meters). It is preferred here to stabilize the solution by setting the gradient term to zero, which gives geoid heights compatible with Helmert orthometric heights. (This DWC-problem is still under investigation. Right now, we believe that the present solution is best in practice.)
- The EGM (in this step 2) is the **satellite-only model GOCO05S** limited to M=240.
- Please note that a -6 mm *zero-degree* term is included in the combined atmospheric correction. This should be discussed by the IHRF WG. Check the derivation.
- Spherical cap: 2 degrees.
- Signal degree variances: Tscherning and Rapp (1974)
- Error degree variances for the EGM: Computed from the published degree-order standard errors for GOCO05S.
- Error degree variances for the surface gravity data (from step 1) are computed as a combination of bandlimited white noise and correlated (colored) noise computed using the reciprocal distance model of Moritz (1980),

  White noise: σ = 1.0 mGal (approximately equal to the standard uncertainties estimated by LSC in Step 1).

  Colored noise: σ = 0.5 mGal with corr. length 0.5° (*chosen to give equal error degree variance of the EGM and of the terrestrial gravity at degree ~150*; cf. Ågren et al. 2009.) The choice was made as it gives a good fit to the old GNSS/levelling at the same time as the passage degree ~150 is believed to be rather realistic.

Selected references

Forsberg R (2003) An Overview Manual for the GRAVSOFT Geodetic Gravity Field Modelling Programs. Draft - 1st edition. Kort & Matrikelstyrelsen, Copenhagen

Heiskanen W A, Moritz H (1967) Physical Geodesy. San Francisco W. H. Freeman and Company.

Moritz H., 1980. Advanced Physical Geodesy. Wichmann

Moritz H (2000) Geodetic Reference System 1980. J Geod 74: 128-133

Sjöberg LE (2000) Topographic effects by the Stokes- Helmert method of geoid and quasigeoid Determination. J Geod 74: 255-268

Sjöberg LE (2003a) A solution to the downward continuation effect on the geoid determined by Stokes' formula. J Geod 77: 94-100

Sjöberg LE (2003b) A computational scheme to model the geoid by the modified Stokes' formula without gravity reductions. J Geod 77: 423-432

Sjöberg LE (2004) A spherical harmonic representation of the ellipsoidal correction to the modified Stokes' formula. J Geod 78: 180-186

Sjöberg LE (2007) The topographic bias by analytical continuation in physical geodesy. J Geod 81: 345-350

Sjöberg LE, Nahavandchi H (2000) The atmospheric effects in Stokes' formula. Geophys. J. Int. 140: 95-100

Tscherning CC, Rapp RH (1974) Closed covariance expressions for gravity anomalies, geoid undulations, and deflections of the vertical implied by anomaly degree variance models. Rep 355, Dept Geod Sci, Ohio State Univ, Columbus

Ågren J (2004) Regional Geoid Determination Methods for the Era of Satellite Gravimetry – Numerical Investigations Using Synthetic Earth Gravity Models. Doctoral Dissertation in Geodesy, TRITA—INFRA 04-033, Royal Institute of Technology (KTH), Stockholm

Ågren J, Sjöberg L E, Kiamehr R (2009) The new gravimetric quasigeoid model KTH08 over Sweden. Journal of Applied Geodesy 3 (3)